# ENERGY SOLUTION TO SCHRÖDINGER-POISSON SYSTEM IN THE TWO-DIMENSIONAL WHOLE SPACE

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Abstract. We consider the Cauchy problem of the two-dimensional Schrödinger-Poisson system in the energy class. Though the Newtonian potential diverges at the spatial infinity in the logarithmic order, global well-posedness is proven in both defocusing and focusing cases. The key is a decomposition of the nonlinearity into a sum of the linear logarithmic potential and a good remainder, which enables us to apply the perturbation method. Our argument can be adapted to the one-dimensional problem.

## 1. Introduction

This paper is devoted to the study of the Schödinger-Poisson system

$$\begin{cases} i\partial_t u + \dfrac{1}{2}\Delta u = \lambda P u, \quad (t,x) \in \mathbb{R}^{1+2}, \\ -\Delta P = |u|^2, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\lambda$ is a real constant. We suppose $P$ is the Newtonian potential

$$P = -\frac{1}{2\pi}(\log|x| * |u|^2) \tag{1.2}$$

where $*$ denotes the convolution. For a suitable $u$, this is the unique strong solution of $-\Delta P = |u|^2$ under the condition

$$|\nabla P| \to 0 \text{ as } |x| \to \infty, \quad \nabla P \in L^\infty(\mathbb{R}^2), \quad P(0) = \int_{\mathbb{R}^2} (\log|y|)|u(y)|^2 dy$$

(see [11]). When the dimensions are larger than two, the Schrödinger-Poisson system is a special case of the Hartree equation and one of the typical example of the nonlinear Schrödinger equation with a nonlocal nonlinearity, and there is large amount of literature (see [6] and references therein). On the other hand, the two-dimensional case is less studied. In [1, 18], (1.1) is considered with some restrictive assumptions such as a neutrality condition which confirms that the Newtonian potential (1.2) does not diverge at the spatial infinity and in particular belongs to $L^2$ space. The Poisson equation is sometimes posed with a background (or doping profile):

$$-\Delta P = |u|^2 - b,$$

where $b$ is a given positive function. Then, the neutrality condition is $\int |u|^2 - b dx = 0$ or equivalently $\mathcal{F}(|u|^2 - b)(0) = 0$. When we consider the problem in dimensions less than three, this condition is useful to control $P$. Notice that this condition excludes all nontrivial solutions when $b \equiv 0$, and that we need to remove this condition for the study of (1.1). In [11], the above

assumptions are removed and the existence of a unique *local* solution is proven for data in the usual Sobolev space $H^s(\mathbb{R}^2)$ $(s>2)$ despite the fact that the nonlinear potential diverges at the spatial infinity. Since (1.2) is not necessarily defined for $u \in H^s$ $(s>2)$ we introduced a new formula

$$P = -\frac{1}{2\pi}\int_{\mathbb{R}^2}\left(\log\frac{|x-y|}{|y|}\right)|u(y)|^2dy$$

which makes sense merely if $|u|^2 \in L^p(\mathbb{R}^2)$ $(p\in(1,2))$. We underline that the local solutions given there do not have finite energy (the energy is given in (1.5) below). Our aim in this paper is to prove that there exists a time-global solution if initial data has finite energy.

For our analysis, the following reduction is crucial: We guess that the Newtonian potential (1.2) may behave like $-\frac{1}{2\pi}\|u\|_{L^2}^2\log|x|$ at the spatial infinity, which will be the bad part of the nonlinearity, and decompose the nonlinearity as

$$\lambda Pu = -\frac{\lambda}{2\pi}\|u\|_{L^2}^2(\log\langle x\rangle)u - \frac{\lambda}{2\pi}u\int_{\mathbb{R}^2}\left(\log\frac{|x-y|}{\langle x\rangle}\right)|u(y)|^2dy,$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$. We then obtain

$$i\partial_t u + \frac{1}{2}\Delta u + \frac{\lambda}{2\pi}\|u\|_{L^2}^2(\log\langle x\rangle)u = -\frac{\lambda}{2\pi}u\int_{\mathbb{R}^2}\left(\log\frac{|x-y|}{\langle x\rangle}\right)|u(y)|^2dy.$$

It will turn out that the bad part of $P$ is correctly extracted from the original nonlinearity and therefore the behavior of the "new nonlinearity" becomes better. Notice that one can also expect that $\|u\|_{L^2}$ is conserved because $\lambda$ is a real number. Hence, putting

$$m := -\frac{\lambda}{2\pi}\|u_0\|_{L^2}^2,$$

we reach to the equation

$$\text{(1.3)}\quad \begin{cases} i\partial_t u + \left(\frac{1}{2}\Delta - m\log\langle x\rangle\right)u = -\frac{\lambda}{2\pi}u\int_{\mathbb{R}^2}\left(\log\frac{|x-y|}{\langle x\rangle}\right)|u(y)|^2dy, \\ u(0,x) = u_0(x). \end{cases}$$

Notice that $-m\log\langle x\rangle$ is now completely independent of $u$ and that it therefore can be regarded as a linear potential. In what follows, we work with this equation. Observe that if there exists a solution to (1.3) conserving $\|u\|_{L^2}$, then it is also a solution of (1.1).

Now, the linear part of the equation is not $i\partial_t + (1/2)\Delta$ but $i\partial_t + (1/2)\Delta - m\log\langle x\rangle$. Thus, a natural choice of the function space on which we shall work is not the Sobolev space $H^1(\mathbb{R}^2)$ any more, but the following one:

$$\text{(1.4)}\quad \begin{aligned} &\mathcal{H} := \{u \in H^1(\mathbb{R}^2); \sqrt{\log\langle x\rangle}u \in L^2\}, \\ &\|u\|_{\mathcal{H}} := \|u\|_{H^1(\mathbb{R}^2)} + \left\|\sqrt{\log\langle\cdot\rangle}u\right\|_{L^2(\mathbb{R}^2)}. \end{aligned}$$

If $m>0$, that is, if $\lambda<0$, then the above space coincides with the form domain of the positive operator $-\frac{1}{2}\Delta + m\log\langle x\rangle$. Our main result is the following:

**Theorem 1.1.** *The problem* (1.3) *is globally well-posed in* $\mathcal{H}$*. Moreover, the solution conserves* $\|u(t)\|_{L^2}$ *and the energy*

$$E(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{\lambda}{4\pi}\int_{\mathbb{R}^2} (\log|x-y|)|u(t,x)|^2|u(t,y)|^2 dxdy. \tag{1.5}$$

**Corollary 1.2.** *The Problem* (1.1) *is globally well-posed in* $\mathcal{H}$*.*

*Remark* 1.3. Let $u \in C(\mathbb{R};\mathcal{H})$ be a solution of (1.3) (and of (1.1)) given in Theorem 1.1. Then, $v := u\exp(-i\frac{\lambda}{2\pi}\int_0^t \|\sqrt{\log|\cdot|}u(s)\|_{L^2}^2 ds)$ solves

$$\begin{cases} i\partial_t v + \frac{1}{2}\Delta v = -\frac{\lambda}{2\pi} v \int_{\mathbb{R}^2} \left(\log\frac{|x-y|}{|y|}\right)|v(y)|^2 dy, \\ v(0,x) = u_0(x). \end{cases} \tag{1.6}$$

Notice that the nonlinearity of (1.6) makes sense without the momentum condition $\sqrt{\log|\cdot|}v \in L^2$. This observation explains why existence of a time-local solution can be proven by assuming only $u_0 \in H^s(\mathbb{R}^2)$ $(s > 1)$ in [11].

1.1. **Consequent results.** Our argument is also applicable to (1.1) involving a power type nonlinearity:

$$\begin{cases} i\partial_t u + \frac{1}{2}\Delta u = \lambda P u + \eta|u|^{p-1}u, \quad (t,x) \in \mathbb{R}^{1+2}, \\ -\Delta P = |u|^2, \\ u(0,x) = u_0(x), \end{cases} \tag{1.7}$$

where $\eta$ is a real number and $p \geqslant 2$.

**Theorem 1.4.** *The problem* (1.7) *is globally well-posed in* $\mathcal{H}$ *if either one of the following conditions is satisfied:*

1. $\eta \geqslant 0$, $\lambda \in \mathbb{R}$ *and* $p \geqslant 2$*;*
2. $\eta < 0$, $\lambda \in \mathbb{R}$, *and* $2 \leqslant p < 3$*;*
3. $\eta < 0$, $\lambda > 0$, $p = 3$, *and* $\|u_0\|_{\mathcal{H}}$ *is small;*
4. $\eta < 0$, $\lambda < 0$, $p \geqslant 3$, *and* $\|u_0\|_{\mathcal{H}}$ *is small.*

*Moreover, the solution conserves* $\|u(t)\|_{L^2}$ *and the energy*

$$\begin{aligned} E_p(t) := &\frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{\lambda}{4\pi}\int_{\mathbb{R}^2}(\log|x-y|)|u(t,x)|^2|u(t,y)|^2 dxdy \\ &+ \frac{\eta}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}. \end{aligned} \tag{1.8}$$

The proof is done with a straight-forward modification (see Section 4). The case where $p = 3$ is known as the $L^2$-critical case. Since the $\mathcal{H}$-norm contains derivative, it seems difficult to treat the case $1 < p < 2$. Nevertheless, we can show global well-posed in a slightly smaller function space $\mathcal{H}^{1,2} := \{u \in H^1(\mathbb{R}^2); u\log\langle x\rangle \in L^2\}$.

**Theorem 1.5.** *Suppose* $1 < p < 2$*. For* $\eta, \lambda \in \mathbb{R}$ *The problem* (1.7) *is globally well-posed in the space* $\mathcal{H}^{1,2}$*. Moreover, the solution conserves* $\|u(t)\|_{L^2}$ *and the energy* $E_p(t)$ *given in* (1.8)*.*

We can also handle the one-dimensional problem

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_{xx} u = -\frac{\lambda}{2}(|x| * |u|^2)u + \eta |u|^{p-1}u, \quad (t,x) \in \mathbb{R}^{1+1}, \\ u(0,x) = u_0(x), \end{cases} \tag{1.9}$$

where $\lambda, \eta \in \mathbb{R}$ and $p \geqslant 2$. The one dimensional problem was studied in [7, 14, 15]. The global well-posedness of (1.9) was shown in the space $\{f \in H^1(\mathbb{R}); |x|f \in L^2(\mathbb{R})\}$ in [15], and in the space $\{f \in H^1(\mathbb{R}); \sqrt{|x|}f \in L^2(\mathbb{R})\}$ with a presence of background in [7], provided $\lambda > 0$ and data is small relative to the background. We can prove the global well-posedness result of (1.9) including these results.

**Theorem 1.6.** *The problem* (1.9) *is globally well-posed in* $\{f \in H^1(\mathbb{R}); \sqrt{|x|}f \in L^2(\mathbb{R})\}$ *if* $\lambda \in \mathbb{R}$ *and either one of the following conditions is satisfied:*

1. $\eta \geqslant 0$, $\lambda \in \mathbb{R}$, *and* $p \geqslant 2$;
2. $\eta < 0$, $\lambda \in \mathbb{R}$, *and* $2 \leqslant p < 5$;
3. $\eta < 0$, $\lambda > 0$, $p = 5$, *and* $\|u_0\|_{H^1} + \|\sqrt{|\cdot|}u_0\|_{L^2}$ *is small;*
4. $\eta < 0$, $\lambda < 0$, $p \geqslant 5$, *and* $\|u_0\|_{H^1} + \|\sqrt{|\cdot|}u_0\|_{L^2}$ *is small.*

*The solution conserves* $\|u\|_{L^2}$ *and the energy*

$$\widetilde{E}(t) := \frac{1}{2}\|\partial_x u\|_{L^2(\mathbb{R})}^2 - \frac{\lambda}{2}\iint_{\mathbb{R}^2} |x-y||u(x)|^2|u(y)|^2 dxdy + \frac{\eta}{p+1}\|u\|_{L^{p+1}(\mathbb{R})}^{p+1}. \tag{1.10}$$

The one-dimensional version of Theorem 1.5 is as follows, which reproduce the same result in [15, Theorem 2.1] when $\eta < 0$ and $\lambda > 0$.

**Theorem 1.7.** *Suppose* $1 < p < 2$. *For* $\eta, \lambda \in \mathbb{R}$ *The problem* (1.9) *is globally well-posed in the space* $\Sigma := \{u \in H^1(\mathbb{R}^2); |x|u \in L^2\}$. *Moreover, the solution conserves* $\|u(t)\|_{L^2}$ *and the energy* $\widetilde{E}(t)$ *given in* (1.10).

As in the two dimensional case, the key is a "reduction" of (1.9) to

$$\begin{cases} i\partial_t u + \frac{1}{2}\partial_{xx} u + \frac{\lambda\|u_0\|_{L^2}^2}{2}|x|u = -\frac{\lambda}{2}u\int_{\mathbb{R}} (|x-y|-|x|)|u(y)|^2 dy + \eta|u|^{p-1}u, \\ u(0,x) = u_0(x). \end{cases}$$

We briefly mention about other related works. Oh considered in [12] the Cauchy problem of the nonlinear Schrödinger equation with general potential and $L^2$-subcritical power-type nonlinearity, and proved global well-posedness in the form domain of $-\frac{1}{2}\Delta + V$, provided the potential $V \geqslant 0$ satisfies $\partial^\alpha V \in L^\infty$ for $|\alpha| \geqslant 2$ (see also [6]). In particular, the case where the potential $V$ is a quadratic polynomial is extensively studied. In this case, we have several special properties such as explicit representations of linear solutions, called Mehler's formula, and/or of the Heisenberg observables. We refer the reader to [2, 3, 4, 10, 19] for $H^1$-subcritical and $H^1$-critical power-type nonlinearity and to [5] for $H^1$-subcritical Hartree type nonlinearity. In [16], the ground states of (1.1) is treated.

The rest of the paper is organized as follows: We collect some basic estimates in Section 2, and, in Section 3 we prove Theorem 1.1. Section 4 is devoted to the study of (1.7).

## 2. Preliminaries

2.1. **Strichartz estimate.** We first summarize the properties on the operator

$$A := \frac{1}{2}\Delta - m \log \langle x \rangle \,, \tag{2.1}$$

where $m \neq 0$ is a real constant. For any $m$, $A$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ (see [13]). Since our potential is sub-quadratic, that is, since $|\partial^\alpha \log \langle x \rangle\,| \to 0$ as $|x| \to \infty$ for $|\alpha| = 2$ and $\partial^\alpha \log \langle x \rangle \in L^\infty$ for $|\alpha| \geqslant 3$, the following estimate is established in [17]: For any $T > 0$,

$$\left\| e^{itA}\varphi \right\|_{L^\infty} \leqslant C|t|^{-1} \left\| \varphi \right\|_{L^1}$$

for $t \in [-T, T]$, where $C$ depends on $T$ (see also [8]). Once we know this type of estimate, the Strichartz estimate follows by interpolation. We say that a pair $(q, r)$ is admissible if $2 \leqslant r < \infty$ and $2/q = \delta(r) := 1 - 2/r$.

**Lemma 2.1** (Strichartz's estimate)**.** *For any $T > 0$, the following properties hold:*

- *Suppose $\varphi \in L^2(\mathbb{R}^2)$. For any admissible pair $(q, r)$, there exists a constant $C = C(T, q, r)$ such that*
$$\left\| e^{itA}\varphi \right\|_{L^q((-T,T);L^r)} \leqslant C \left\| \varphi \right\|_{L^2} .$$
- *Let $I \subset (-T, T)$ be an interval and $t_0 \in \overline{I}$. For any admissible pairs $(q, r)$ and $(\gamma, \rho)$, there exists a constant $C = C(t, q, r, \gamma, \rho)$ such that*
$$\left\| \int_{t_0}^{t} e^{i(t-s)A} F(s) ds \right\|_{L^q(I;L^r)} \leqslant C \left\| F \right\|_{L^{\gamma'}(I;L^{\rho'})}$$
*for every $F \in L^{\gamma'}(I; L^{\rho'})$.*

2.2. **Some estiamtes.**

**Lemma 2.2.** *Let $W$ be an arbitrary weight function such that $\nabla W$, $\Delta W \in L^\infty(\mathbb{R}^2)$. It holds for all $T > 0$, admissible pair $(q, r)$, and $\varphi \in \mathcal{H}$ that*

$$\begin{aligned}
\left\| [\nabla, e^{itA}]\varphi \right\|_{L^q((-T,T);L^r)} &\leqslant C|T| \left\| \varphi \right\|_2 , \\
\left\| [W, e^{itA}]\varphi \right\|_{L^q((-T,T);L^r)} &\leqslant C|T| \left\| (1 + \nabla)\varphi \right\|_2 .
\end{aligned}$$

*Proof.* Since $v = e^{itA}\varphi$ solves $i\partial_t v + Av = 0$, an explicit calculation shows

$$[\nabla, e^{itA}]\varphi = -i \int_0^t e^{i(t-s)A} \frac{mx}{1 + x^2} e^{isA}\varphi ds$$

and

$$[W, e^{itA}]\varphi = i \int_0^t e^{i(t-s)A} \left( \nabla W \cdot \nabla + \frac{1}{2}\Delta W \right) e^{isA}\varphi ds.$$

The Strichartz estimate therefore gives the desired estimates. □

The following is useful for estimates of the nonlinearity in (1.3).

**Lemma 2.3.** *Set a function*

$$K(x,y)=\frac{\log\frac{|x-y|}{\langle x\rangle}}{1+\log\langle y\rangle}$$

*of* $x,y\in\mathbb{R}^2$. *For any* $p\in[1,\infty)$ *and* $\varepsilon>0$, *there exist a function* $W(x,y)\geqslant 0$ *with* $\|W\|_{L^\infty_y L^p_x}\leqslant\varepsilon$ *and a constant* $C_0$ *such that*

$$|K(x,y)|\leqslant C_0+W(x,y)$$

*holds for all* $(x,y)\in\mathbb{R}^{2+2}$.

*Proof.* Take $\eta\in(0,1]$ and set $W(x,y)=|K(x,y)|\mathbf{1}_{|x-y|\leqslant\eta}(x,y)$. If $\eta$ is sufficiently small then

$$\|W(\cdot,y)\|_{L^p}\leqslant\frac{\|\log|x|\|_{L^p(|x|\leqslant\eta)}+\log\langle|y|+\eta\rangle\,\|1\|_{L^p(|x|\leqslant\eta)}}{1+\log\langle y\rangle}\leqslant\varepsilon$$

since $\log|x|$ belongs to $L^p_{\mathrm{loc}}(\mathbb{R}^2)$ for all $p<\infty$. Moreover, by (2.12) of [11],

$$\sup_{|x-y|\geqslant\eta}K(x,y)\leqslant 1+\log\frac{\sqrt{3}}{\eta}$$

for any $\eta\leqslant 1$, which completes the proof. □

*Remark* 2.4. In 1D case, the corresponding estimate is

$$\left\|\frac{|x-y|-|x|}{1+|y|}\right\|_{L^\infty_{x,y}(\mathbb{R}^2)}\leqslant 1.$$

## 3. Proof of the theorem

### 3.1. **Local well-posedness.**

**Lemma 3.1.** *Let* $(q_0,r_0)$ *be an admissible pair with* $r_0>2$. *For any* $u_0\in\mathcal{H}$, *there exist an existence time* $T=T(\|u_0\|_{\mathcal{H}})$ *and a unique solution* $u\in C((-T,T);\mathcal{H})\cap L^{q_0}((-T,T);L^{r_0})\cap C^1((-T,T);\mathcal{H}^*)$. *The solution conserves* $\|u(t)\|_{L^2}$ *and the energy* (1.5). *Moreover, the map* $u_0\mapsto u$ *is continuous from* $\mathcal{H}$ *to* $C((-T,T);\mathcal{H})$.

*Proof.* We write $L^p((-T,T);X)=L^p_TX$, for short. Define a Banach space

$$\mathcal{H}_{T,M}:=\{f\in L^\infty((-T,T);\mathcal{H});\|f\|_{\mathcal{H}_T}\leqslant M\}$$

with norm

$$\|f\|_{\mathcal{H}_T}:=\|f\|_{L^\infty_T\mathcal{H}}+\|f\|_{L^{q_0}_TW^{1,r_0}}+\left\|\sqrt{\log\langle x\rangle}f\right\|_{L^{q_0}_TL^{r_0}}.$$

We show that if $r_0>2$ then there exist $M=M(\|u_0\|_{\mathcal{H}})$ and $T=T(\|u_0\|_{\mathcal{H}})$ such that

$$\begin{aligned}Q[u](t,x)&:=(e^{itA}u_0)(x)\\&\quad+\frac{i}{2\pi}\left(\int_0^t e^{i(t-s)A}\left(\int_{\mathbb{R}^2}\log\frac{|\cdot-y|}{\langle\cdot\rangle}|u(s,y)|^2dy\right)u(s,\cdot)ds\right)(x)\end{aligned}$$

becomes a contraction map from $\mathcal{H}_{T,M}$ to itself, where $A$ is defined in (2.1).

Set
$$K(x,y) = \frac{\log \frac{|x-y|}{\langle x \rangle}}{1+\log \langle y \rangle}.$$
Then, by Lemma 2.3, there exist a nonnegative function $W \in L^\infty_y L^{r_0'}_x$ and a constant $C_0$ such that
$$|K(x,y)| \leqslant C_0 + W(x,y).$$
Recall that $r_0 \in (2,\infty)$ and so $r_0' := r_0/(r_0-1) \in (1,2)$. We hence see that
$$Pu = \iint K(x,y)(1+\log\langle y\rangle)|u(y)|^2 u(x) dy\, dx$$
satisfies
$$\|Pu\|_{L^2} \leqslant C(\|u\|_{L^2} + \|u\|_{L^{r_0}}) \left\| \sqrt{1+\log\langle x \rangle} u \right\|_{L^2}^2 .$$
Take $L^1_T$ norm to yield

(3.1)
$$\|Pu\|_{L^1_T L^2} \leqslant C(T \|u\|_{L^\infty_T L^2} + T^{\frac12+\frac1{r_0}} \|u\|_{L^{q_0}_T L^{r_0}}) \left\| \sqrt{1+\log\langle x \rangle} u \right\|_{L^\infty_T L^2}^2 .$$
By the Strichartz estimate, we end up with
$$\|Q[u]\|_{L^\infty_T L^2} + \|Q[u]\|_{L^{q_0}_T L^{r_0}} \leqslant C\|u_0\|_{L^2} + C(T+T^{\frac12+\frac1{r_0}}) \|u\|^3_{\mathcal{H}_T} . \tag{3.2}$$
We next estimate $\nabla Q[u]$. One easily sees that
$$\begin{aligned}\nabla Q[u] = {}& e^{itA}\nabla u_0 - i\int_0^t e^{i(t-s)A}\nabla(Pu)(s)ds \\ &+ [\nabla, e^{itA}]u_0 - i\int_0^t [\nabla, e^{i(t-s)A}](Pu)(s)ds.\end{aligned}$$
We deduce from Lemma 2.2 with $(q,r)=(\infty,2)$ that
$$\int_0^t \left\| [\nabla, e^{i(t-s)A}](Pu)(s) \right\|_{L^2} ds \leqslant \int_0^t (t-s)\|Pu(s)\|_{L^2} ds \leqslant |t| \, \|Pu\|_{L^1_T L^2} .$$
The right hand side is bounded as in (3.1). $[\nabla, e^{itA}]u_0$ is handled similarly. Mimicking (3.1), we infer that

(3.3)
$$\|P\nabla u\|_{L^1_T L^2} \leqslant C(T\|\nabla u\|_{L^\infty_T L^2} + T^{\frac12+\frac1{r_0}} \|\nabla u\|_{L^{q_0}_T L^{r_0}}) \left\| \sqrt{1+\log\langle x\rangle} u \right\|^2_{L^\infty_T L^2}$$
Now, let us estimate $(\nabla P)u$. It writes
$$(\nabla P)(x) u(x) = \left( \int_{\mathbb{R}^2} \left( \frac{x-y}{|x-y|^2} - \frac{x}{1+x^2} \right) |u(y)|^2 dy \right) u(x),$$
and so
$$\begin{aligned}\|(\nabla P)u\|_{L^2} &\leqslant C \left\| (|x|^{-1} * |u|^2) + \langle\cdot\rangle^{-1} \|u\|^2_{L^2} \right\|_{L^{\frac{2r_0}{r_0-2}}} \|u\|_{L^{r_0}} \\ &\leqslant C(\|u\|^2_{L^{\frac{2r_0}{r_0-1}}} + \|u\|^2_{L^2}) \|u\|_{L^{r_0}} \\ &\leqslant C(\|u\|^2_{L^2} + \|\nabla u\|^2_{L^2}) \|u\|_{L^{r_0}}\end{aligned}$$

by the Hardy-Littlewood-Sobolev and the Sobolev inequalities. We see that

$$\|(\nabla P)u\|_{L^1_T L^2} \leqslant C(T\|u\|^2_{L^\infty_T L^2} + T^{\frac{1}{2}+\frac{1}{r_0}}\|u\|^2_{L^{q_0}_T L^{r_0}})\|u\|_{L^\infty_T L^2}. \tag{3.4}$$

We deduce from the Strichartz estimate that

$$\|\nabla Q[u]\|_{L^\infty_T L^2} + \|\nabla Q[u]\|_{L^{q_0}_T L^{r_0}} \leqslant C\|\nabla u_0\|_{\mathcal{H}} + C(T + T^{\frac{1}{2}+\frac{1}{r_0}})\|u\|^3_{\mathcal{H}_T}. \tag{3.5}$$

Let us proceed to the estimate of $\sqrt{\log\langle x\rangle}Q[u]$. It holds that

$$\sqrt{1+\log\langle x\rangle}Q[u] = e^{itA}\sqrt{1+\log\langle x\rangle}u_0 - i\int_0^t e^{i(t-s)A}\sqrt{1+\log\langle x\rangle}Pu(s)ds + R,$$

where

$$R = [\sqrt{1+\log\langle x\rangle}, e^{itA}]u_0 - i\int_0^t [\sqrt{1+\log\langle x\rangle}, e^{i(t-s)A}]Pu(s)ds.$$

A use of Lemma 2.2 with $W = \sqrt{1+\log\langle x\rangle}$ yields

$$\begin{aligned}\|R\|_{L^\infty_T L^2} + \|R\|_{L^{q_0}_T L^{r_0}} &\leqslant CT\|u_0\|_{\mathcal{H}} + CT\|(1+\nabla)(Pu)\|_{L^1_T L^2} \\ &\leqslant CT\|u_0\|_{\mathcal{H}} + CT(T + T^{\frac{1}{2}+\frac{1}{r_0}})\|u\|^3_{\mathcal{H}_T}\end{aligned}$$

where we have used (3.1), (3.3), and (3.4). As in (3.1), it holds that

$$\begin{aligned}\|P(Wu)\|_{L^1_T L^2} &\leqslant C(T\|Wu\|_{L^\infty_T L^2} + T^{\frac{1}{2}+\frac{1}{r_0}}\|Wu\|_{L^{q_0}_T L^{r_0}})\|Wu\|^2_{L^\infty_T L^2} \\ &\leqslant C(T + T^{\frac{1}{2}+\frac{1}{r_0}})\|u\|^3_{\mathcal{H}_T},\end{aligned}$$

where $W = \sqrt{1+\log\langle x\rangle}$. We conclude from the Strichartz estimate, (3.2), and (3.5) that

$$\|Q[u]\|_{\mathcal{H}_T} \leqslant C_1\|u_0\|_{\mathcal{H}} + C_2(T + T^{\frac{1}{2}+\frac{1}{r_0}})\|u\|^3_{\mathcal{H}_T}.$$

A similar argument shows

$$\|Q[u_1] - Q[u_2]\|_{\mathcal{H}_T} \leqslant C_3(T + T^{\frac{1}{2}+\frac{1}{r_0}})(\|u_1\|_{\mathcal{H}_T} + \|u_2\|_{\mathcal{H}_T})^2\|u_1 - u_2\|_{\mathcal{H}_T}.$$

Thus, if we take $M \geqslant 2C_1\|u_0\|_{\mathcal{H}}$ then there exists $T = T(M)$ such that $Q$ is a contraction map from $\mathcal{H}_{T,M}$ to itself.

The conservations of $\|u(t)\|_{L^2}$ is shown by multiplying (1.3) by $\overline{u}$ and integrating the imaginary part. To prove the energy conservation, we need a regularizing argument. Note that (1.3) can be solved also in the space $\{f \in H^2(\mathbb{R}^2) : \log\langle x\rangle f \in L^2\}$, which is one of dense subsets of $\mathcal{H}$, in an essentially same way. We omit details. □

3.2. **Global existence.** We first give a useful blow-up criteria.

**Lemma 3.2.** *Suppose $u_0 \in \mathcal{H}$. Let $u \in C((-T_{\min}, T_{\max}); \mathcal{H})$ be a unique maximal solution given by Lemma 3.1. If $T_{\max} < \infty$ (resp. $T_{\min} < \infty$), then $\|\nabla u(t)\|_{L^2} \to \infty$ as $t \uparrow T_{\max}$ (resp. $t \downarrow -T_{\min}$).*

*Proof.* We only consider positive time. Suppose $T_{\max} < \infty$. Then, $\|u(t)\|_{\mathcal{H}}$ has to diverge as $t \uparrow T_{\max}$. Otherwise, we can extend the solution beyond $T_{\max}$ by Lemma 3.1. Recall that $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$. Since

$$\begin{aligned}\frac{d}{dt}\left\|\sqrt{\log\langle x\rangle}u(t)\right\|_{L^2}^2 &= 2\operatorname{Re}\int(\log\langle x\rangle)\partial_t u(t)\overline{u(t)}dx\\ &= -\operatorname{Im}\int(\log\langle x\rangle)\Delta u(t)\overline{u(t)}dx\\ &= \operatorname{Im}\int\frac{x}{1+x^2}\cdot\nabla u(t)\overline{u(t)}dx,\end{aligned}$$

it holds that

$$\left\|\sqrt{\log\langle x\rangle}u(t_2)\right\|_{L^2}^2 \leqslant \left\|\sqrt{\log\langle x\rangle}u(t_1)\right\|_{L^2}^2 + |t_2-t_1|\,\|\nabla u\|_{L^\infty((t_1,t_2);L^2)}\|u_0\|_{L^2}$$

for all $-T_{\min} < t_1 < t_2 < T_{\max}$. This implies that if we assume

$$\limsup_{t\uparrow T_{\max}}\|\nabla u(t)\|_{L^2} < \infty$$

then $\|u(t)\|_{\mathcal{H}}$ never blows up. We hence obtain the lemma. □

*Remark* 3.3. As in [9], the solution breaks down with concentration at a point if $\|\sqrt{\log\langle x\rangle}u(t)\|_{L^2} = 0$. However, this does not occur when $\|\nabla u(t)\|$ is bounded above. Indeed, since

$$\|u\|_{L^2(|x|<r)} \leqslant \|1\|_{L^4(|x|<r)}\|u\|_{L^4} \leqslant Cr^{\frac{1}{2}}\|\nabla u\|_{L^2}^{\frac{1}{2}}$$

for any $r > 0$ and since

$$\|u\|_{L^2(|x|<r)} = \|u_0\|_{L^2} - \|u\|_{L^2(|x|\geqslant r)} \geqslant \|u_0\|_{L^2} - \frac{\|\sqrt{\log\langle x\rangle}u\|_{L^2}}{(\log\langle r\rangle)^{1/2}},$$

by letting $r = \|\sqrt{\log\langle x\rangle}u\|_{L^2}$, we obtain

$$\|\sqrt{\log\langle x\rangle}u\|_{L^2}^{-\frac{1}{2}} \leqslant C\left(\frac{\|\sqrt{\log\langle x\rangle}u\|_{L^2}}{\log\left\langle\|\sqrt{\log\langle x\rangle}u\|_{L^2}\right\rangle}\right)^{\frac{1}{2}} + C\|\nabla u\|_{L^2}^{\frac{1}{2}},$$

which implies $\|\sqrt{\log\langle x\rangle}u\|_{L^2}$ is strictly positive if $\|\nabla u\|_{L^2} < \infty$.

*Proof of Theorem 1.1.* Let us establish a priori estimate of $\|\nabla u(t)\|_{L^2}$.

We first consider the case $\lambda < 0$. Since $\log|x| \geqslant 0$ for $|x| \geqslant 1$,

$$\begin{aligned}&-\frac{\lambda}{4\pi}\iint_{\mathbb{R}^{2+2}}\log|x-y||u(x)|^2|u(y)|^2dxdy\\ &\geqslant -\frac{|\lambda|}{4\pi}\iint_{|x-y|<1}|\log|x-y||\,|u(x)|^2|u(y)|^2dxdy\\ &\geqslant -\frac{|\lambda|}{4\pi}\|\log|x|\|_{L^2(|x|\leqslant 1)}^2\|u\|_{L^4}^2\|u\|_{L^2}^2\end{aligned}$$

By the $L^2$-conservation and the Sobolev embedding, we have

$$\|\nabla u(t)\|_{L^2}^2 \leqslant 2E_0 + C\|\nabla u(t)\|_{L^2}. \tag{3.6}$$

Therefore, there exists a constant $M$ independent of $t$ such that $\|\nabla u(t)\|_{L^2} \leqslant M$.

We now suppose $\lambda > 0$. By Lemma 2.3, for any $\varepsilon > 0$ there exists a constant $C_0$ such that the following estimate holds:

$$
\begin{aligned}
&\frac{\lambda}{4\pi}\iint_{\mathbb{R}^{2+2}} \log|x-y||u(x)|^2|u(y)|^2dxdy \\
&\leqslant \frac{\lambda}{4\pi}\iint_{\mathbb{R}^{2+2}} \log\frac{|x-y|}{\langle x\rangle}|u(x)|^2|u(y)|^2dxdy + \frac{\lambda}{4\pi}\|u_0\|_{L^2}^2\left\|\sqrt{\log\langle x\rangle}u\right\|_{L^2}^2 \\
&\leqslant \frac{\lambda}{4\pi}(C_0\|u_0\|_{L^2}^2+\varepsilon\|u\|_{L^4}^2)\left\|\sqrt{1+\log\langle x\rangle}u\right\|_{L^2}^2 + \frac{\lambda}{4\pi}\|u_0\|_{L^2}^2\left\|\sqrt{\log\langle x\rangle}u\right\|_{L^2}^2 \\
&\leqslant \frac{\lambda C_0}{4\pi}\|u_0\|_{L^2}^4 + \frac{\lambda(C_0+1)}{4\pi}\|u_0\|_{L^2}^2\left\|\sqrt{\log\langle x\rangle}u\right\|_{L^2}^2 + C\varepsilon\|u_0\|_{L^2}^3\|\nabla u\|_{L^2} \\
&\quad + C\varepsilon\|u_0\|_{L^2}\|\nabla u\|_{L^2}\left\|\sqrt{\log\langle x\rangle}u\right\|_{L^2}^2 \\
&\leqslant C_1 + C_2(\varepsilon+|t|)\sup_{s\in[0,t]}\|\nabla u(s)\|_{L^2} + C_3\varepsilon|t|\sup_{s\in[0,t]}\|\nabla u(s)\|_{L^2}^2,
\end{aligned}
$$

where $C_i$ $(i=1,2,3)$ depends only on $\lambda$, $C_0$, $\|u_0\|_{\mathcal{H}}$, and $\varepsilon$. Fix $T>0$. Taking $\varepsilon < (8C_3T)^{-1}$, we deduce from the conservation of $E(t)$ that

$$
\left(\sup_{s\in[0,t]}\|\nabla u(s)\|_{L^2}\right)^2 \leqslant 4E(0) + 4C_1 + 4C_2(\varepsilon+2T)\sup_{s\in[0,t]}\|\nabla u(s)\|_{L^2} \tag{3.7}
$$

for $0\leqslant t\leqslant 2T$. This implies that

$$
\sup_{t\in[0,2T]}\|\nabla u(t)\|_{L^2} \leqslant C(\|u_0\|_{\mathcal{H}}, T) < \infty.
$$

Since $T$ is arbitrary, we obtain the global existence. □

## 4. Remarks on the problem with power nonlinearity

We give a rough sketch of the proofs of Theorem 1.4 and 1.5 in this section.

*Proof of Theorem 1.4.* The local well-posedness part holds if $p\geqslant 2$ as in the proof of Lemma 3.1. The restriction $p\geqslant 2$ is required when we estimate

$$
\begin{aligned}
|\nabla(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2)| &\leqslant C_p(|u_1|^{p-2}+|u_2|^{p-2})(|\nabla u_1|+|\nabla u_2|)|u_1-u_2| \\
&\quad + C_p(|u_1|^{p-1}+|u_2|^{p-1})|\nabla(u_1-u_2)|.
\end{aligned}
$$

By exactly the same argument as in Lemma 3.2, the problem of global existence boils down to obtaining an a priori bound of $\|\nabla u(t)\|_{L^2}$. Recall that the conserved energy is

$$
\begin{aligned}
E_p(t) := \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 &- \frac{\lambda}{4\pi}\int_{\mathbb{R}^2}(\log|x-y|)|u(t,x)|^2|u(t,y)|^2dxdy \\
&+ \frac{\eta}{p+1}\|u(t)\|_{L^{p+1}}^{p+1}.
\end{aligned}
$$

*The case $\eta>0$.* We have

$$
\|\nabla u(t)\|_{L^2}^2 \leqslant E_p(t) + \frac{\lambda}{4\pi}\int_{\mathbb{R}^2}(\log|x-y|)|u(t,x)|^2|u(t,y)|^2dxdy.
$$

By the same argument as in the case $\eta=0$, we prove global existence.

*The case $\eta < 0$ and $\lambda < 0$.* Since

$$\frac{|\eta|}{p+1}\|u(t)\|_{L^{p+1}}^{p+1} \leqslant C_{\eta,p}\|u_0\|_{L^2}^2\|\nabla u(t)\|_{L^2}^{p-1},$$

we obtain

$$\|\nabla u(t)\|_{L^2}^2 \leqslant 2E_0 + C\|\nabla u(t)\|_{L^2} + 2C_{\eta,p}\|u_0\|_{L^2}^2\|\nabla u(t)\|_{L^2}^{p-1}$$

as in (3.6). Uniform bound of $\|\nabla u(t)\|_{L^2}$ is then obtained either the case $p < 3$ or the case $p \geqslant 3$ and $\|u_0\|_{L^2}$ is small.

*The case $\eta < 0$ and $\lambda > 0$.* As in (3.7), for any $T > 0$, there exist $\varepsilon$, $C_1$, and $C_2$ such that

$$\begin{aligned}\left(\sup_{s\in[0,t]}\|\nabla u(s)\|_{L^2}\right)^2 &\leqslant 4E(0) + 4C_1 + 4C_2(\varepsilon + 2T)\sup_{s\in[0,t]}\|\nabla u(s)\|_{L^2}\\ &\quad + 4C_{\eta,p}\|u_0\|_{L^2}^2\left(\sup_{s\in[0,t]}\|\nabla u(s)\|_{L^2}\right)^{p-1}.\end{aligned}$$

for $t \leqslant 2T$. Therefore, if $p < 3$ or if $p = 3$ and $\|u_0\|_{L^2}$ is small, we obtain

$$\sup_{t\in[0,2T]}\|\nabla u(t)\|_{L^2} \leqslant C(\|u_0\|_{\mathcal{H}}, T) < \infty.$$

This concludes the proof of Theorem 1.4. □

*Proof of Theorem 1.5.* We denote $L^p((-T,T);X) = L^p_T X$. Our strategy for local well-posedness is to use the contraction argument in a complete metric space $(\mathcal{H}^{1,2}_{T,M}, d)$, where

$$\begin{aligned}\mathcal{H}^{1,2}_{T,M} &:= \{f \in C((-T,T);H^1); \|f\|_{\mathcal{H}^{1,2}_T} \leqslant M\},\\ \|f\|_{\mathcal{H}^{1,2}_T} &:= \|f\|_{L^\infty_T\mathcal{H}} + \|f\|_{L^{q_0}_T W^{1,r_0}} + \|f\log\langle x\rangle\|_{L^{q_0}_T L^{r_0}}\end{aligned}$$

for an admissible pair $(q_0, r_0)$ with $r_0 > 2$, and the metric $d$ is given by

$$d(f,g) = \|f-g\|_{L^\infty_T L^2} + \|f-g\|_{L^{q_0}_T L^{r_0}}. \tag{4.1}$$

We shall show

$$\begin{aligned}Q[u](t,x) &:= (e^{itA}u_0)(x)\\ &\quad + \frac{i}{2\pi}\left(\int_0^t e^{i(t-s)A}\left(\int_{\mathbb{R}^2}\log\frac{|\cdot - y|}{\langle\cdot\rangle}|u(s,y)|^2dy\right)u(s,\cdot)ds\right)(x)\\ &\quad - i\eta\left(\int_0^t e^{i(t-s)A}(|u|^{p-1}u)(s)ds\right)(x)\end{aligned}$$

is a contraction map in $(\mathcal{H}^{1,2}_{T,M}, d)$. Mimicking the proof of Lemma 3.1, one shows that for any $M > 0$, there exists $T > 0$ such that $Q : \mathcal{H}^{1,2}_{T,M} \to \mathcal{H}^{1,2}_{T,M}$. To prove $Q$ is a contraction with respect to the metric $d$, the following

estimate is crucial:

$$\begin{aligned}
&\left\| \left( \int_{\mathbb{R}^2} \log \frac{|x-y|}{\langle x \rangle} |u_1(y)|^2 dy \right) u_1 - \left( \int_{\mathbb{R}^2} \log \frac{|x-y|}{\langle x \rangle} |u_2(y)|^2 dy \right) u_2 \right\|_{L^2} \\
&\leqslant \left\| \left( \int_{\mathbb{R}^2} \log \frac{|x-y|}{\langle x \rangle} |u_1(y)|^2 dy \right) (u_1 - u_2) \right\|_{L^2} \\
&\quad + \left\| \left( \int_{\mathbb{R}^2} \log \frac{|x-y|}{\langle x \rangle} (|u_1(y)|^2 - |u_2(y)|^2) dy \right) u_2 \right\|_{L^2} \\
&\leqslant C(\|u_1\|_{L^2}^2 + \|\sqrt{\log \langle x \rangle} u_1\|_{L^2}^2)(\|u_1 - u_2\|_{L^2} + \|u_1 - u_2\|_{L^{r_0}}) \\
&\quad + C(\left\| |u_1|^2 - |u_2|^2 \right\|_{L^1} + \|(|u_1|^2 - |u_2|^2) \log \langle x \rangle \|_{L^1})(\|u_2\|_{L^2} + \|u_2\|_{L^{r_0}}) \\
&\leqslant C(\|u_1\|_{L^2}^2 + \|\sqrt{\log \langle x \rangle} u_1\|_{L^2}^2)(\|u_1 - u_2\|_{L^2} + \|u_1 - u_2\|_{L^{r_0}}) \\
&\quad + C(\|u_1\|_{L^2} + \|u_2\|_{L^2} + \|u_1 \log \langle x \rangle \|_{L^2} + \|u_2 \log \langle x \rangle \|_{L^2}) \\
&\qquad \times (\|u_2\|_{L^2} + \|u_2\|_{L^{r_0}}) \|u_1 - u_2\|_{L^2} .
\end{aligned}$$

By the Strichartz estimate, letting $T$ smaller if necessary, we hence obtain

$$d(Q[u_1], Q[u_2]) \leqslant \frac{1}{2} d(u_1, u_2)$$

for any $u_1, u_2 \in \mathcal{H}^{1,2}_{T,M}$.

A similar result as Lemma 3.2 holds since

$$\left| \frac{d}{dt} \|\log \langle x \rangle u(t)\|_{L^2}^2 \right| = \left| \operatorname{Im} \int \frac{2x \log \langle x \rangle}{1+x^2} \cdot \nabla u(t) \overline{u(t)} dx \right| \leqslant C \|\nabla u(t)\|_{L^2} .$$

Now, we have a priori bound of $\|\nabla u(t)\|_{L^2}$ as in the case $2 \leqslant p < 3$ of Theorem 1.4, which proves the global well-posedness. $\square$

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